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Shear Deformable Theories for Cylindrical Laminates—Equilibrium and Buckling with Applications

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A higher order theory is developed (kinematic relations, constitutive relations, equilibrium equations, and boundary conditions), which includes initial geometric imperfections and transverse shear effects for a laminated cylindrical configuration under the actions of lateral pressure, axial compression, and eccentrically applied torsion. Equilibrium equations and boundary conditions are presented for the higher order shear deformation theory. The perturbation technique is employed and buckling equations are derived for symmetric laminates under the action of uniform axial compression and lateral pressure. Buckling loads are computed for axially compressed cylindrical laminates of finite length and pressure loaded very long configurations.

I. Introduction

SHELL theories and formulations are inherently approximate in nature because they are developed on the basis of assumptions and hypotheses. Certain of these assumptions are made in order to reduce a three-dimensional problem to a two-dimensional one, such as the Kirchhoff-Love hypotheses of straight inextensional normals, while others deal with the degree of nonlinearity needed to evaluate accurately rotations, changes in curvature, and in-plane shear and extensional strains in terms of reference surface displacement components.

It is well accepted that the classical two-dimensional shell theory, as applied to laminated composite construction, leads to fairly accurate estimates for laminate behavior (displacements, stresses, strains, buckling loads, vibration frequencies, etc.) of thin (ratio of total thickness to smallest initial radius of curvature <<1) laminates. There are several reasons for removing some of the classical shell theory assumptions, especially as applied to fibrous composite construction.

First of all, most of the present-day advanced composites have a low transverse shear modulus and, therefore, transverse shear deformation plays a much more important role in the kinematics of composite laminated shells than in metallic ones. Furthermore, in order to achieve the same strength as metals (with a considerable saving in weight), in some applications we need to use thicker walls that in turn reduce the ratio of the smallest initial radius of curvature to total thickness. This, in many instances, pushes us to the boundary between thin shells and thick shells.

If we must still use two-dimensional theory (as opposed to the more complex three-dimensional, nonlinear anisotropic elasticity theory) for predicting the response of composite laminated shells, then we must remove the classical assumption of "normals remain normal" and account for the shear deformations in connection with the transverse direction. Thus, it is important to remove several of the simplifying assumptions in the classical two-dimensional shell theory and develop a higher order theory for analyzing multilayer composite shell configurations. The main emphasis is on the effect of all these approximations on the buckling response of laminated shells.

Efforts to account for the effect of shear deformations or even normal strains have shown up very recently. Noor and Burton[1] review the existing shear deformation theories on multilayered composite plates. Librescu and Stein[2] and Reddy[3,4] report refined nonlinear analyses accounting for transverse shear deformations, concentrating primarily to flat configurations. Srinivas[5] performed an analysis of laminated, composite, cylindrical shells with general boundary conditions using three methods of approaching the problem. In one method, he considers the effects of transverse shear deformation and transverse normal stress. Nonlinear analysis of orthotropic, laminated shells of revolution with transverse shear deformation is presented by Eldridge.[6] Reddy and Liu[7,8] developed a higher order shear deformation theory that corresponds to a cubic variation in the normal direction for the in-plane displacement components and to a constant (function of in-plane coordinates only) for the transverse displacement. They included a von Karman type of kinematic nonlinearity for the reference surface extensional strains (lowest order of nonlinearity applicable to shallow shell configurations—Donnell type of approximation), developed equations of motion and finite element models. In Ref. 7, they present applications to cylindrical and spherical shells by employing the linearized version of their equations. Barbero et al.[9] developed a general two-dimensional theory for laminated cylindrical shells. They presented Navier-type solutions of the linear theory for simply supported boundary conditions, and their theory accounts for a desired degree of approximation of the displacements through the thickness. Dennis and Palazotto[10] made a finite element model for orthotropic cylindrical pressure vessels using a higher order shear deformable theory. Another effort was made by Khdeir et al.[11] who developed a shear deformable theory of laminated composite shallow shell-type panels. Whitney and Sun[12] have developed a higher order refined laminated plate theory applicable to fiber reinforced composite materials under impact loading. The effect of shear deformation on postbuckling behavior of laminated beams was examined by Sheinman and Adan.[13] Stein[14] performed a nonlinear analysis for plates and shells including the effects of transverse shearing and normal strains. Zukas[15] and Vinson and Waltz[16] performed analyses, accounting for the effect of normal and shear strains in laminated shells, by using linearized strain-displacement relations, which are applicable to small deflection shell response (small deflection theories).

Moreover, McCarty[17] developed a three-dimensional, nonlinear dynamic finite element model for the response of a thick laminated shell to impact loads. Bert[18] analyzed the small-amplitude free vibrations of thick, circular cylindrical shells, using shear deformable theory. One more effort is

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reported by Sciuva and Carrera,[19] on the stability of moderately thick laminated anisotropic and sandwich panels accounting for the effect of shear. Elishakoff et al.[20] studied the dynamic response of transverse shear deformable laminated shells subjected to random excitation. They checked the response of cylindrical circular shells, spherical shells, and cylindrical panels by using a first-order transverse shear deformation theory. Finally, Donnell[21] developed three equilibrium equations in terms of three displacements in scalar mathematical form for thick general shells under general loading.

The present work deals with the development of the kinematic relations, equilibrium equations and related boundary conditions, and buckling equations and related boundary conditions, for a laminated, cylindrical, moderately thick shell, including initial geometric imperfections and transverse shear effects. A higher order displacement field in the thickness direction is employed. Finally, buckling applications are presented for thick, axially loaded and pressure-loaded, laminated, cylindrical shells with fixed ends.

II. Mathematical Formulation

The cylindrical shell is assumed to be relatively thick, geometrically imperfect and to have a laminated construction that is generally asymmetric (general stacking of the orthotropic laminae). The loading includes uniform lateral pressure, axial compression, and eccentric torsion. The Kirchhoff-Love hypotheses do not apply, and three assumptions are made in this work: 1) the laminae are orthotropic; 2) transverse shear strains exist; and 3) the material behavior is linearly elastic.

A. Kinematic Relations

1. Higher Order Shear Deformation Theory

The three-dimensional displacement components are u, v, w in the axial (x), circumferential (y), and radial (along the thickness) (z) directions, respectively. A general cubic variation in the z coordinate is assumed for u and v, whereas a constant in the z coordinate is assumed for w (see Refs. 4, 7, 8, and 14).

$$u(x, y, z) = \bar{u}(x, y) + z\Psi_x(x, y) + z^2\xi_x(x, y) + z^3\zeta_x(x, y)$$

$$v(x, y, z) = \bar{v}(x, y) + z\Psi_y(x, y) + z^2\xi_y(x, y) + z^3\zeta_y(x, y)$$

$$w(x, y, z) = \bar{w}(x, y) \qquad (1)$$

where $\bar{u}(x, y)$, $\bar{v}(x, y)$, $\bar{w}(x, y)$ are the reference surface displacements, and Ψ_x, ξ_x, ζ_x, Ψ_y, ξ_y, ζ_y are some functions of position on the reference surface (x, y), which appear due to the expansions of the three-dimensional displacements in the z-direction. From the engineering nonlinear expressions for strains, by performing a power series expansion in the z-direction, and by retaining terms up to second degree, new expressions for the strains are obtained. For the purposes of this work, concentration to problems where the rotations $\{(\bar{w}_{,x} + w^0_{,x}), (\bar{w}_{,y} + w^0_{,y} - [\bar{v}/(R + z)]\}$ are moderately small is made, according to Ref. 14. Note that $w^0(x, y)$ denotes the initial geometric imperfection of the reference surface and R the shell radius. Thus, only products containing these terms are retained and the remaining products are neglected as small by comparison. Substitutions of the displacement relation into the strain expressions, application of the above simplifications, and knowledge of zero shear tractions on the upper and lower surfaces (equivalent to shear strains being zero) lead to the following set of kinematic relations (for details, see Ref. 22):

$$\varepsilon_{xx} = \bar{u}_{,x} + z\Psi_{x,x} - \frac{4z^3}{3h^2}(\bar{w}_{,xx} + w^0_{,xx} + \Psi_{x,x}) + \frac{1}{2}(\bar{w}_{,x})^2 + \bar{w}_{,x}w^0_{,x} \qquad (2)$$

$$\begin{aligned}\varepsilon_{\theta\theta} = {} & \frac{1}{R+z}(\bar{w} + \bar{v}_{,\theta} + z\Psi_{y,\theta}) + \frac{4Rz^2}{(R+z)(16h^2R^2 - 48R^4 - h^4)}\Big[(h^2 - 4R^2) \\ & \cdot\left(\Psi_{y,\theta} + \frac{\bar{w}_{,\theta\theta} + w^0_{,\theta\theta} - \bar{v}_{,\theta}}{R}\right)\Big] + \frac{z^3}{(R+z)(16h^2R^2 - 48R^4 - h^4)h^2} \\ & \cdot\left[(64R^4 - 16R^2h^2)\left(\Psi_{y,\theta} + \frac{\bar{w}_{,\theta\theta} + w^0_{,\theta\theta} - \bar{v}_{,\theta}}{R}\right)\right] + \frac{1}{2}\left(\frac{\bar{w}_{,\theta} - \bar{v}}{R+z}\right)^2 + \frac{w^0_{,\theta}}{(R+z)^2}(\bar{w}_{,\theta} - \bar{v})\end{aligned} \qquad (3)$$

$$\varepsilon_{zz} = 0 \qquad (4)$$

$$\begin{aligned}\gamma_{x\theta} = {} & \frac{1}{R+z}\left[u_{,\theta} + z\Psi_{x,\theta} - \frac{4z^3}{3h^2}(\bar{w}_{,x\theta} + w^0_{,x\theta} + \Psi_{x,\theta})\right] + \bar{v}_{,x} + z\Psi_{y,x} + \frac{4Rz^2}{(16h^2R^2 - 48R^4 - h^4)} \\ & \cdot\left[(h^2 - 4R^2)\left(\Psi_{y,x} + \frac{\bar{w}_{,x\theta} + w^0_{,x\theta} - \bar{v}_{,x}}{R}\right)\right] + \frac{z^3}{(16h^2R^2 - 48R^4 - h^4)h^2}\Big[(64R^4 - 16R^2h^2) \\ & \cdot\left(\Psi_{y,x} + \frac{\bar{w}_{,x\theta} + w^0_{,x\theta} - \bar{v}_{,x}}{R}\right)\Big] + \frac{1}{R+z}(\bar{w}_{,x} + w^0_{,x})(\bar{w}_{,\theta} + w^0_{,\theta} - \bar{v})\end{aligned} \qquad (5)$$

$$\gamma_{xz} = \bar{w}_{,x} + w^0_{,x} + \Psi_x - \frac{4z^2}{h^2}(\bar{w}_{,x} + w^0_{,x} + \Psi_x) \qquad (6)$$

$$\begin{aligned}\gamma_{\theta x} = {} & \Psi_y + \left(2z - \frac{z^2}{R+z}\right)\frac{4R}{(16h^2R^2 - 48R^4 - h^4)} \cdot\left[(h^2 - 4R^2)\left(\Psi_y + \frac{\bar{w}_{,\theta} + w^0_{,\theta} - \bar{v}}{R}\right)\right] \\ & + \left(3z^2 - \frac{z^3}{R+z}\right)\frac{1}{(16h^2R^2 - 48R^4 - h^4)h^2} \cdot\left[(64R^4 - 16R^2h^2)\left(\Psi_y + \frac{\bar{w}_{,\theta} + w^0_{,\theta} - \bar{v}}{R}\right)\right] \\ & + \frac{1}{R+z}[\bar{w}_{,\theta} + w^0_{,\theta} - \bar{v} + z(-\Psi_y)]\end{aligned} \qquad (7)$$

2. First Order and Classical Theories

The Kinematic relations corresponding to the first order shear deformation theory are as follows (for details, see Ref. 22):

$$\varepsilon_{xx} = \bar{u}_{,x} + z\Psi_{x,x} + \tfrac{1}{2}(\bar{w}_{,x})^2 + \bar{w}_{,x}w^0_{,x}$$

$$\varepsilon_{\theta\theta} = \frac{1}{R+z}\left[\bar{w} + \bar{v}_{,\theta} + z\Psi_{y,\theta} + \frac{1}{2(R+z)}(\bar{w}_{,\theta} - \bar{v})^2 + \frac{w^0_{,\theta}}{R+z}(\bar{w}_{,\theta} - \bar{v})\right]$$

$$\varepsilon_{zz} = 0$$

$$\gamma_{x\theta} = \frac{1}{R+z}[(\bar{u}_{,\theta} + z\Psi_{x,\theta} + (\bar{w}_{,x} + w^0_{,x})(\bar{w}_{,\theta} + w^0_{,\theta} - \bar{v})] + \bar{v}_{,x} + z\Psi_{y,x}$$

$$\gamma_{xz} = \bar{w}_{,x} + w^0_{,x} + \Psi_x$$

$$\gamma_{\theta z} = \Psi_y + \frac{1}{R+z}(\bar{w}_{,\theta} + w^0_{,\theta} - \bar{v} - z\Psi_y) \tag{8}$$

The corresponding "classical theory" kinematic relations are given below. They correspond to Sanders' type[23] of kinematic relations with moderate rotations about in-plane axes and negligible rotations about the normal:

$$\varepsilon_{xx} = \bar{u}_{,x} + \tfrac{1}{2}(\bar{w}_{,x})^2 + \bar{w}_{,x}w^0_{,x} - z(\bar{w}_{,xx} + w^0_{,xx})$$

$$\varepsilon_{\theta\theta} = \frac{1}{R}\left[\bar{w} + \bar{v}_{,\theta} - \frac{z}{R}(\bar{w}_{,\theta\theta} + w^0_{,\theta\theta} - \bar{v}_{,\theta}) + \frac{1}{2R}(\bar{w}_{,\theta} - \bar{v})^2 + \frac{w^0_{,\theta}}{R}(\bar{w}_{,\theta} - \bar{v})\right]$$

$$\gamma_{x\theta} = \frac{1}{R}[\bar{u}_{,\theta} + (\bar{w}_{,x} + w^0_{,x})(\bar{w}_{,\theta} + w^0_{,\theta} - \bar{v}) - z(\bar{w}_{,x\theta} + w^0_{,x\theta})] + \bar{v}_{,x} - \frac{z}{R}(\bar{w}_{,x\theta} + w^0_{,x\theta} - \bar{v}_{,x})$$

$$\varepsilon_{zz} = \gamma_{z\theta} = \gamma_{zx} = 0 \tag{9}$$

B. Constitutive Equations

The equations relating the stresses to the strains for a lamina of a laminate, in terms of structural axes, x, y, z, parameters are given by the following:

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{\theta\theta}\\ \sigma_{zz}\\ \sigma_{\theta x}\\ \sigma_{xz}\\ \sigma_{x\theta}\end{Bmatrix} = \begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{13} & 0 & 0 & \bar{Q}_{16}\\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{23} & 0 & 0 & \bar{Q}_{26}\\ \bar{Q}_{13} & \bar{Q}_{23} & \bar{Q}_{33} & 0 & 0 & \bar{Q}_{36}\\ 0 & 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} & 0\\ 0 & 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55} & 0\\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{36} & 0 & 0 & \bar{Q}_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{\theta\theta}\\ \varepsilon_{zz}\\ \gamma_{\theta z}\\ \gamma_{xz}\\ \gamma_{x\theta}\end{Bmatrix} \tag{10}$$

where the "bar" stiffnesses are given in terms of the orthotropic stiffnesses, Q_{ij}, through the usual transformation equations (see Ref. 24). Moreover, the orthotropic stiffnesses in terms of the engineering constants, E_i, ν_{ij}, and G_{ij} ($i, j = 1, 2, 3$) may also be found in Ref. 24. For the present work, the simpler form suggested in Ref. 16 is employed in relating the orthotropic stiffnesses to the engineering constants.

C. Governing Equations

Use of the principle of the stationary value of the total potential yields the equilibrium equations and associated boundary conditions. Only those corresponding to the higher order theory are shown herein:

due to $\delta\bar{u}$

$$RN_{xx,x} + M_{xx,x} + N_{x\theta,\theta} = 0 \tag{11}$$

due to $\delta\bar{v}$

$$-N_{\theta\theta,\theta} + \frac{\Delta_2}{\Delta_1}K_{\theta\theta,\theta} + \frac{\Delta_3}{R\Delta_1}L_{\theta\theta,\theta} - P_{\theta\theta}(\bar{w}_{,\theta} + w^0_{,\theta} - \bar{v}) - RN_{x\theta,x} - M_{x\theta,x} + \frac{R\Delta_2}{\Delta_1}K_{x\theta,x} + \frac{\Delta_2+\Delta_3}{\Delta_1}L_{x\theta,x} + \frac{\Delta_3}{R\Delta_1}Q_{x\theta,x}$$
$$- N_{\theta z} - \frac{2R\Delta_2}{\Delta_1}M_{\theta z} - \frac{\Delta_2 + 3\Delta_3}{\Delta_1}K_{\theta z} - \frac{2\Delta_3}{R\Delta_1}L_{\theta z} - N_{x\theta}(\bar{w}_{,x} + w^0_{,x}) = 0 \tag{12}$$

due to $\delta\bar{w}$

$$-\frac{4R}{3h^2}L_{xx,xx} - \frac{4}{3h^2}Q_{xx,xx} - R[N_{xx}(\bar{w}_{,x} + w^0_{,x})]_{,x} + \frac{\Delta_2}{\Delta_1}K_{\theta\theta,\theta\theta} + N_{\theta\theta} - [M_{xx}(\bar{w}_{,x} + w^0_{,x})]_{,x} + \frac{\Delta_3}{R\Delta_1}L_{\theta\theta,\theta\theta}$$
$$+ \frac{R\Delta_2}{\Delta_1}K_{x\theta,x\theta} - [P_{\theta\theta}(\bar{w}_{,\theta} + w^0_{,\theta} - \bar{v})]_{,\theta} + \left(\frac{\Delta_2+\Delta_3}{\Delta_1} - \frac{4}{3h^2}\right)L_{x\theta,x\theta} + \frac{\Delta_3}{R\Delta_1}Q_{x\theta,x\theta}$$
$$- [N_{x\theta}(\bar{w}_{,\theta} + w^0_{,\theta} - \bar{v})]_{,x} - [N_{x\theta}(\bar{w}_{,x} + w^0_{,x})]_{,\theta} - RN_{xz,x} - M_{xz,x} + \frac{4R}{h^2}K_{xz,x} + \frac{4}{h^2}L_{xz,x}$$
$$- \frac{2R\Delta_2}{\Delta_1}M_{\theta z,\theta} - \frac{\Delta_2 + 3\Delta_3}{\Delta_1}K_{\theta z,\theta} - N_{\theta z,\theta} - \frac{2\Delta_3}{R\Delta_1}L_{\theta z,\theta} = A_1 \tag{13}$$

where $A_1 = -p_o(R + h/2) - p_i(R - h/2)$ for the case of lateral pressure, and p_o is the pressure on the outer surface and p_i on the inner:

due to $\delta\Psi_x$

$$-RM_{xx,x} + \frac{4R}{3h^2}L_{xx,x} - K_{xx,x} + \frac{4}{3h^2}Q_{xx,x} - M_{x\theta,\theta} + \frac{4}{3h^2}L_{x\theta,\theta} + RN_{xz} - \frac{4R}{h^2}K_{xz} + M_{xz} - \frac{4}{h^2}L_{xz} = 0 \tag{14}$$

due to $\delta\Psi_y$

$$-M_{\theta\theta,\theta} - \frac{R\Delta_2}{\Delta_1}K_{\theta\theta,\theta} - \frac{\Delta_3}{\Delta_1}L_{\theta\theta,\theta} - RM_{x\theta,x} - \left(1 + \frac{R^2\Delta_2}{\Delta_1}\right)K_{x\theta,x} - \frac{R(\Delta_2+\Delta_3)}{\Delta_1}L_{x\theta,x}$$
$$- \frac{\Delta_3}{\Delta_1}Q_{x\theta,x} + RN_{\theta z} + \frac{2R^2\Delta_2}{\Delta_1}M_{\theta z} + \frac{R(\Delta_2 + 3\Delta_3)}{\Delta_1}K_{\theta z} + \frac{2\Delta_3}{\Delta_1}L_{\theta z} = 0 \tag{15}$$

where

$$N_{ij} = \int_{-h/2}^{h/2}\sigma_{ij}\,dz; \qquad M_{ij} = \int_{-h/2}^{h/2}\sigma_{ij}z\,dz$$

$$K_{ij} = \int_{-h/2}^{h/2}\sigma_{ij}z^2\,dz; \qquad L_{ij} = \int_{-h/2}^{h/2}\sigma_{ij}z^3\,dz$$

$$Q_{ij} = \int_{-h/2}^{h/2}\sigma_{ij}z^4\,dz; \qquad P_{\theta\theta} = \int_{-h/2}^{h/2}\frac{\sigma_{\theta\theta}}{R+z}\,dz \tag{16}$$

and

$$\Delta_1 = h^2(16h^2R^2 - 48R^4 - h^4); \qquad \Delta_2 = 4h^2(h^2 - 4R^2)$$
$$\Delta_3 = 64R^4 - 16R^2h^2; \qquad \Delta_4 = 2(12R^2 - 3h^2)h^2$$
$$\Delta_5 = 2h^4 - 8R^2h^2$$

The corresponding boundary conditions are obtained from the following set.

At $\theta = \theta_a, \theta_b$, the following quantities are prescribed:

EITHER	OR
$N_{x\theta}$	$\delta\bar{u}$
$N_{\theta\theta} - \frac{\Delta_2}{\Delta_1}K_{\theta\theta} - \frac{\Delta_3}{R\Delta_1}L_{\theta\theta}$	$\delta\bar{v}$
$-\frac{\Delta_2}{\Delta_1}K_{\theta\theta,\theta} - \frac{\Delta_3}{R\Delta_1}L_{\theta\theta,\theta} + P_{\theta\theta}(\bar{w}_{,\theta} + w^0{}_{,\theta} - \bar{v}) + N_{x\theta}(\bar{w}_{,x} + w^0{}_{,x}) + N_{\theta z} + \frac{2R\Delta_2}{\Delta_1}M_{\theta z} + \frac{\Delta_2 + 3\Delta_3}{\Delta_1}K_{\theta z} + \frac{2\Delta_3}{R\Delta_1}L_{\theta z}$	$\delta\bar{w}$
$M_{x\theta} - \frac{4}{3h^2}L_{x\theta}$	$\delta\Psi_x$
$M_{\theta\theta} + \frac{R\Delta_2}{\Delta_1}K_{\theta\theta} + \frac{\Delta_3}{\Delta_1}L_{\theta\theta}$	$\delta\Psi_y$
$\frac{\Delta_2}{\Delta_1}K_{\theta\theta} + \frac{\Delta_3}{R\Delta_1}L_{\theta\theta}$	$\delta\bar{w}_{,\theta}$

At $x = 0, L$, the following quantities are prescribed:

EITHER	OR
$RN_{xx} + M_{xx} + \delta_1 R\bar{N}_{xx}$	$\delta\bar{u}$
$RN_{x\theta} + M_{x\theta} - \frac{R\Delta_2}{\Delta_1}K_{x\theta} - \frac{\Delta_2 + \Delta_3}{\Delta_1}L_{x\theta} - \frac{\Delta_3}{R\Delta_1}Q_{x\theta} - \delta_2 R\bar{N}_{xy}$	$\delta\bar{v}$
$\frac{4R}{3h^2}L_{xx,x} + \frac{4}{3h^2}Q_{xx,x} + (RN_{xx} + M_{xx})(\bar{w}_{,x} + w^0{}_{,x}) - \frac{R\Delta_2}{\Delta_1}K_{x\theta,\theta} - \left(\frac{\Delta_2 + \Delta_3}{\Delta_1} - \frac{4}{3h^2}\right)L_{x\theta,\theta} - \frac{\Delta_3}{R\Delta_1}Q_{x\theta,\theta} + N_{x\theta}(\bar{w}_{,\theta} + w^0{}_{,\theta} - \bar{v}) + RN_{xz} - \frac{4R}{h^2}K_{xz} + M_{xz} - \frac{4}{h^2}L_{xz} - \frac{R\Delta_2}{\Delta_1}K_{x\theta,x} - \left(\frac{\Delta_2 + \Delta_3}{\Delta_1} - \frac{4}{3h^2}\right)L_{x\theta,x} - \frac{\Delta_3}{R\Delta_1}Q_{x\theta,x}$	$\delta\bar{w}$
$RM_{xx} - \frac{4R}{3h^2}L_{xx} + K_{xx} - \frac{4}{3h^2}Q_{xx}$	$\delta\Psi_x$
$RM_{x\theta} + \left(1 + \frac{R^2\Delta_2}{\Delta_1}\right)K_{x\theta} + \frac{R(\Delta_2 + \Delta_3)}{\Delta_1}L_{x\theta} + \frac{\Delta_3}{\Delta_1}Q_{x\theta}$	$\delta\Psi_y$
$-\frac{4R}{3h^2}L_{xx} - \frac{4}{3h^2}Q_{xx}$	$\delta\bar{w}_{,x}$

where $\delta_1 = 0$, $\delta_2 = 0$ for lateral pressure, $\delta_1 = 1$, $\delta_2 = 0$ for axial compression, $\delta_1 = 1$, $\delta_2 = 0$ for hydrostatic pressure, $\delta_1 = 0$, $\delta_2 = 1$ for shear.

At the corners: $(x = 0, \theta = \theta_a)$, $(x = L, \theta = \theta_a)$, $(x = 0, \theta = \theta_b)$, $(x = L, \theta = \theta_b)$, the following quantities are prescribed:

EITHER	OR
$\frac{R\Delta_2}{\Delta_1}K_{x\theta} + \left(\frac{\Delta_2 + \Delta_3}{\Delta_1} - \frac{4}{3h^2}\right)L_{x\theta} + \frac{\Delta_3}{R\Delta_1}Q_{x\theta}$	$\delta\bar{w}$

D. Buckling Equations

Once the equilibrium equations and the related boundary conditions are derived, the next step is to proceed with the derivation of the buckling equations. This work will eventually specialize to the case of configurations free of initial geometric imperfections and having symmetric laminate construction. These two conditions create a geometry that possesses a momentless prebuckling state and bifurcational buckling from this state to a bent (buckled) state is possible. The perturbation technique (Refs. 25–27) can then be applied to produce the buckling equations and the related boundary conditions. Buckling equations are presented for the case of pressure and/or axial compression.

1. Prebuckling State

As the pressure and the axial compression loads are applied statically, a membrane primary state exists. The primary state is characterized by $M_{xx} = M_{x\theta} = M_{\theta\theta} = 0$. The primary state solution is found to be the following:

$$\bar{u}^{(0)} = Ax, \qquad \bar{v}^{(0)} = 0, \qquad \bar{w}^{(0)} = C$$
$$\Psi_x^{(0)} = 0, \qquad \Psi_y^{(0)} = 0 \tag{17}$$

where the constants A and C are given by:

$$A = [K_1(RT_{12} + AA_{12}) - K_2T_{22}]/[A_{12}(RT_{12} + AA_{12}) - T_{22}(RA_{11} + B_{11})]$$
$$C = [A_{12}K_2 - (RA_{11} + B_{11})K_1]/[A_{12}(RT_{12} + AA_{12}) - T_{22}(RA_{11} + B_{11})] \tag{18}$$

where A_{ij}, B_{ij}, T_{ij}, etc. are shell stiffness parameters defined below, p_o and p_i denote normal tractions (pressures) on the outer and inner shell surfaces, respectively, and $\bar{N}_{xx}$ denotes the uniform compressive stress resultant:

$$R_{ij} = \int_{-h/2}^{h/2} \bar{Q}_{ij}(R + z)^{-2}\,dz \qquad T_{ij} = \int_{-h/2}^{h/2} Q_{ij}(R + z)^{-1}\,dz$$
$$A_{ij} = \int_{-h/2}^{h/2} \bar{Q}_{ij}\,dz \qquad B_{ij} = \int_{-h/2}^{h/2} \bar{Q}_{ij}z\,dz$$
$$D_{ij} = \int_{-h/2}^{h/2} \bar{Q}_{ij}z^2\,dz \qquad E_{ij} = \int_{-h/2}^{h/2} \bar{Q}_{ij}z^3\,dz$$
$$F_{ij} = \int_{-h/2}^{h/2} \bar{Q}_{ij}z^4\,dz \qquad H_{ij} = \int_{-h/2}^{h/2} \bar{Q}_{ij}z^5\,dz$$
$$I_{ij} = \int_{-h/2}^{h/2} \bar{Q}_{ij}z^6\,dz \qquad O_{ij} = \int_{-h/2}^{h/2} \bar{Q}_{ij}z^7\,dz$$
$$W_{ij} = \int_{-h/2}^{h/2} \bar{Q}_{ij}(R + z)^{-3}\,dz \qquad TT_{ij} = \int_{-h/2}^{h/2} \bar{Q}_{ij}(R + z)^{-2}z\,dz$$
$$AA_{ij} = \int_{-h/2}^{h/2} \bar{Q}_{ij}(R + z)^{-1}z\,dz$$

$$BB_{ij} = \int_{-h/2}^{h/2} \overline{Q}_{ij}(R + z)^{-1}z^2\, dz$$

$$DD_{ij} = \int_{-h/2}^{h/2} \overline{Q}_{ij}(R + z)^{-1}z^3\, dz$$

$$EE_{ij} = \int_{-h/2}^{h/2} \overline{Q}_{ij}(R + z)^{-1}z^4\, dz$$

$$AAA_{ij} = \int_{-h/2}^{h/2} \overline{Q}_{ij}(R + z)^{-2}z^2\, dz$$

$$BBB_{ij} = \int_{-h/2}^{h/2} \overline{Q}_{ij}(R + Z)^{-2}z^3\, dz$$

$$DDD_{ij} = \int_{-h/2}^{h/2} \overline{Q}_{ij}(R + z)^{-2}z^4\, dz \tag{19}$$

2. Buckling State

Once the prebuckling state solution is known, and the perturbation technique is applied, the buckling equations can be derived, by assuming that we can pass from the primary to the bent state through extremely small additional parameters (displacements, stresses, stress resultants, etc.) so that linearization is possible.

If we let $\overline{u}$, $\overline{v}$, $\overline{w}$, Ψ_x, Ψ_y and their derivatives be the small perturbations, and N_{ij}, M_{ij}, K_{ij}, L_{ij}, Q_{ij}, and $P_{\theta\theta}$ be the corresponding perturbed quantities, the buckling equations are:

$$RN_{xx,x} + M_{xx,x} + N_{x\theta,\theta} = 0 \tag{20}$$

$$\begin{aligned} &-N_{\theta\theta,\theta} + \frac{\Delta_2}{\Delta_1}K_{\theta\theta,\theta} + \frac{\Delta_3}{R\Delta_1}L_{\theta\theta,\theta} \\ &- (T_{12}A + R_{22}C)(\overline{w},_\theta - \overline{v}) - RN_{x\theta,x} - M_{x\theta,x} \\ &+ \frac{R\Delta_2}{\Delta_1}K_{x\theta,x} + \frac{\Delta_2 + \Delta_3}{\Delta_1}L_{x\theta,x} + \frac{\Delta_3}{R\Delta_1}Q_{x\theta,x} \\ &- N_{\theta z} - \frac{2R\Delta_2}{\Delta_1}M_{\theta z} - \frac{\Delta_2 + 3\Delta_3}{\Delta_1}K_{\theta z} \\ &- \frac{2\Delta_3}{R\Delta_1}L_{\theta z} - \overline{w},_x(A_{16}A + T_{26}C) = 0 \end{aligned} \tag{21}$$

$$\begin{aligned} &-\frac{4R}{3h^2}L_{xx,xx} - \frac{4}{3h^2}Q_{xx,xx} - R\overline{w},_{xx}(A_{11}A + T_{12}C) \\ &+ \frac{\Delta_2}{\Delta_1}K_{\theta\theta,\theta\theta} + N_{\theta\theta} - \overline{w},_{xx}(B_{11}A + AA_{12}C) \\ &+ \frac{\Delta_3}{R\Delta_1}L_{\theta\theta,\theta\theta} + \frac{R\Delta_2}{\Delta_1}K_{x\theta,x\theta} - (\overline{w},_{\theta\theta} - \overline{v},_\theta) \\ &\times (T_{12}A + R_{22}C) + \left(\frac{\Delta_2 + \Delta_3}{\Delta_1} - \frac{4}{3h^2}\right)L_{x\theta,x\theta} \\ &+ \frac{\Delta_3}{R\Delta_1}Q_{x\theta,x\theta} - (2\overline{w},_{x\theta} - \overline{v},_x)(A_{16}A + T_{25}C) \\ &- RN_{xx,x} - M_{xz,x} + \frac{4R}{h^2}K_{xz,x} + \frac{4}{h^2}L_{xz,x} \\ &- \frac{2R\Delta_2}{\Delta_1}M_{\theta z,\theta} - \frac{\Delta_2 + 3\Delta_3}{\Delta_1}K_{\theta z,\theta} - N_{\theta z,\theta} \\ &- \frac{2\Delta_3}{R\Delta_1}L_{\theta z,\theta} = 0 \end{aligned} \tag{22}$$

$$\begin{aligned} &-RM_{xx,x} + \frac{4R}{3h^2}L_{xx,x} - K_{xx,x} + \frac{4}{3h^2}Q_{xx,x} \\ &- M_{x\theta,\theta} + \frac{4}{3h^2}L_{x\theta,\theta} + RN_{xz} - \frac{4R}{h^2}K_{xz} \\ &+ M_{xz} - \frac{4}{h^2}L_{xz} = 0 \end{aligned} \tag{23}$$

$$\begin{aligned} &-M_{\theta\theta,\theta} - \frac{R\Delta_2}{\Delta_1}K_{\theta\theta,\theta} - \frac{\Delta_3}{\Delta_1}L_{\theta\theta,\theta} - RM_{x\theta,x} \\ &- \left(1 + \frac{R^2\Delta_2}{\Delta_1}\right)K_{x\theta,x} - \frac{R(\Delta_2 + \Delta_3)}{\Delta_1}L_{x\theta,x} \\ &- \frac{\Delta_3}{\Delta_1}Q_{x\theta,x} + RN_{\theta z} + \frac{2R^2\Delta_2}{\Delta_1}M_{\theta z} \\ &+ \frac{R(\Delta_2 + 3\Delta_3)}{\Delta_1}K_{\theta z} + \frac{2\Delta_3}{\Delta_1}L_{\theta z} = 0 \end{aligned} \tag{24}$$

The associated boundary conditions are obtained from the set below.

At $\theta = \theta_a, \theta_b$, the following quantities are zero:

EITHER	OR
$N_{x\theta}$	$\delta\overline{u}$
$N_{\theta\theta} - \frac{\Delta_2}{\Delta_1}K_{\theta\theta} - \frac{\Delta_3}{R\Delta_1}L_{\theta\theta}$	$\delta\overline{v}$
$-\frac{\Delta_2}{\Delta_1}K_{\theta\theta,\theta} - \frac{\Delta_3}{R\Delta_1}L_{\theta\theta,\theta} + (T_{12}A + R_{22}C)(\overline{w},_\theta + \overline{v}) + \overline{w},_x(A_{16}A + T_{26}C) + N_{\theta z} + \frac{2R\Delta_2}{\Delta_1}M_{\theta z} + \frac{\Delta_2 + 3\Delta_3}{\Delta_1}K_{\theta z} + \frac{2\Delta_3}{R\Delta_1}L_{\theta z}$	$\delta\overline{w}$
$M_{x\theta} - \frac{4}{3h^2}L_{x\theta}$	$\delta\Psi_x$
$M_{\theta\theta} + \frac{R\Delta_2}{\Delta_1}K_{\theta\theta} + \frac{\Delta_3}{\Delta_1}L_{\theta\theta}$	$\delta\Psi_y$
$\frac{\Delta_2}{\Delta_1}K_{\theta\theta} + \frac{\Delta_3}{R\Delta_1}L_{\theta\theta}$	$\delta\overline{w},_\theta$

At $x = 0, L$, the following quantities are zero:

EITHER	OR
$RN_{xx} + M_{xx}$	$\delta\overline{u}$
$RN_{x\theta} + M_{x\theta} - \frac{R\Delta_2}{\Delta_1}K_{x\theta} - \frac{\Delta_2 + \Delta_3}{\Delta_1}L_{x\theta} - \frac{\Delta_3}{R\Delta_1}Q_{x\theta}$	$\delta\overline{v}$
$\frac{4R}{3h^2}L_{xx,x} + \frac{4}{3h^2}Q_{xx,x} + \overline{w},_x[(RA_{11} + B_{11})A + (RT_{12} + AA_{12})C] - \frac{R\Delta_2}{\Delta_1}K_{x\theta,\theta} - \left(\frac{\Delta_2 + \Delta_3}{\Delta_1} - \frac{4}{3h^2}\right)L_{x\theta,\theta} - \frac{\Delta_3}{R\Delta_1}Q_{x\theta,\theta}$	

$$+ (A_{16}A + T_{26}C)(\bar{w}_{,\theta} - \bar{v}) + RN_{xz} - \frac{4R}{h^2} K_{xz} + M_{xz} - \frac{4}{h^2} L_{xz} - \frac{R\Delta_2}{\Delta_1} K_{x\theta,x} - \left(\frac{\Delta_2 + \Delta_3}{\Delta_1} - \frac{4}{3h^2}\right) L_{x\theta,x} - \frac{\Delta_3}{R\Delta_1} Q_{x\theta,x} \qquad \delta\bar{w}$$

$$RM_{xx} - \frac{4R}{3h^2} L_{xx} + K_{xx} - \frac{4}{3h^2} Q_{xx} \qquad \delta\Psi_x$$

$$RM_{x\theta} + \left(1 + \frac{R^2\Delta_2}{\Delta_1}\right)K_{x\theta} + \frac{R(\Delta_2 + \Delta_3)}{\Delta_1} L_{x\theta} + \frac{\Delta_3}{\Delta_1} Q_{x\theta} \qquad \delta\Psi_y$$

$$-\frac{4R}{3h^2} L_{xx} - \frac{4}{3h^2} Q_{xx} \qquad \delta\bar{w}_{,x}$$

At the corners: $(x = 0, \theta = \theta_a)$, $(x = L, \theta = \theta_a)$, $(x = 0, \theta = \theta_b)$, $(x = L, \theta = \theta_b)$, the following quantities are zero:

EITHER OR

$$\frac{R\Delta_2}{\Delta_1} K_{x\theta} + \left(\frac{\Delta_2 + \Delta_3}{\Delta_1} - \frac{4}{3h^2}\right)L_{x\theta} + \frac{\Delta_3}{R\Delta_1} Q_{x\theta} \qquad \delta\bar{w}$$

where the small additional integrated stresses are still defined by Eqs. (16) and are related to the kinematics by:

$$N_{xx} = A_{11}\varepsilon_{xx}^0 + B_{11}k_{xx}^1 + D_{11}k_{xx}^2 + E_{11}k_{xx}^3 + R_{12}k_{\theta\theta}^{-2} + T_{12}k_{\theta\theta}^{-1} + A_{12}\varepsilon_{\theta\theta}^0 + B_{12}k_{\theta\theta}^1 + D_{12}k_{\theta\theta}^2 + T_{16}k_{x\theta}^{-1} + A_{16}\gamma_{x\theta}^0 + B_{16}k_{x\theta}^1 + D_{16}k_{x\theta}^2 + E_{16}k_{x\theta}^3$$

$$N_{\theta\theta} = A_{12}\varepsilon_{xx}^0 + B_{12}k_{xx}^1 + D_{12}k_{xx}^2 + E_{12}k_{xx}^3 + R_{22}k_{\theta\theta}^{-2} + T_{22}k_{\theta\theta}^{-1} + A_{22}\varepsilon_{\theta\theta}^0 + B_{22}k_{\theta\theta}^1 + D_{22}k_{\theta\theta}^2 + T_{26}k_{x\theta}^{-1} + A_{26}\gamma_{x\theta}^0 + B_{26}k_{x\theta}^1 + D_{26}k_{x\theta}^2 + E_{26}k_{x\theta}^3$$

$$N_{\theta z} = T_{44}k_{\theta z}^{-1} + A_{44}\gamma_{\theta z}^0 + B_{44}k_{\theta z}^1 + D_{44}k_{\theta z}^2 + A_{45}\gamma_{xz}^0 + D_{45}k_{xz}^2$$

$$N_{xz} = T_{45}k_{\theta z}^{-1} + A_{45}\gamma_{\theta z}^0 + B_{45}k_{\theta z}^1 + D_{45}k_{\theta z}^2 + A_{55}\gamma_{xz}^0 + D_{55}k_{xz}^2$$

$$N_{x\theta} = A_{16}\varepsilon_{xx}^0 + B_{16}k_{xx}^1 + D_{16}k_{xx}^2 + E_{16}k_{xx}^3 + R_{26}k_{\theta\theta}^{-2} + T_{26}k_{\theta\theta}^{-1} + A_{26}\varepsilon_{\theta\theta}^0 + B_{26}k_{\theta\theta}^1 + D_{26}k_{\theta\theta}^2 + T_{66}k_{x\theta}^{-1} + A_{66}\gamma_{x\theta}^0 + B_{66}k_{x\theta}^1 + D_{66}k_{x\theta}^2 + E_{66}k_{x\theta}^3$$

$$P_{\theta\theta} = T_{12}\varepsilon_{xx}^0 + AA_{12}k_{xx}^1 + BB_{12}k_{xx}^2 + DD_{12}k_{xx}^3 + W_{22}k_{\theta\theta}^{-2} + R_{22}k_{\theta\theta}^{-1} + T_{22}\varepsilon_{\theta\theta}^0 + AA_{22}k_{\theta\theta}^1 + BB_{22}k_{\theta\theta}^2 + R_{26}k_{x\theta}^{-1} + T_{26}\gamma_{x\theta}^0 + AA_{26}k_{x\theta}^1 + BB_{26}k_{x\theta}^2 + DD_{26}k_{x\theta}^3$$

$$M_{xx} = B_{11}\varepsilon_{xx}^0 + D_{11}k_{xx}^1 + E_{11}k_{xx}^2 + F_{11}k_{xx}^3 + TT_{12}k_{\theta\theta}^{-2} + AA_{12}k_{\theta\theta}^{-1} + B_{12}\varepsilon_{\theta\theta}^0 + D_{12}k_{\theta\theta}^1 + E_{12}k_{\theta\theta}^2 + AA_{16}k_{x\theta}^{-1} + B_{16}\gamma_{x\theta}^0 + D_{16}k_{x\theta}^1 + E_{16}k_{x\theta}^2 + F_{16}k_{x\theta}^3$$

$$M_{\theta\theta} = B_{12}\varepsilon_{xx}^0 + D_{12}k_{xx}^1 + E_{12}k_{xx}^2 + F_{12}k_{xx}^3 + TT_{22}k_{\theta\theta}^{-2} + AA_{22}k_{\theta\theta}^{-1} + B_{22}\varepsilon_{\theta\theta}^0 + D_{22}k_{\theta\theta}^1 + E_{22}k_{\theta\theta}^2 + AA_{26}k_{x\theta}^{-1} + B_{26}\gamma_{x\theta}^0 + D_{26}k_{x\theta}^1 + E_{26}k_{x\theta}^2 + F_{26}k_{x\theta}^3$$

$$M_{\theta z} = AA_{44}k_{\theta z}^{-1} + B_{44}\gamma_{\theta z}^0 + D_{44}k_{\theta z}^1 + E_{44}k_{\theta z}^2 + B_{45}\gamma_{xz}^0 + E_{45}k_{xz}^2$$

$$M_{xz} = AA_{45}k_{\theta z}^{-1} + B_{45}\gamma_{\theta z}^0 + D_{45}k_{\theta z}^1 + E_{45}k_{\theta z}^2 + B_{55}\gamma_{xz}^0 + E_{55}k_{xz}^2$$

$$M_{x\theta} = B_{16}\varepsilon_{xx}^0 + D_{16}k_{xx}^1 + E_{16}k_{xx}^2 + F_{16}k_{xx}^3 + TT_{26}k_{\theta\theta}^{-2} + AA_{26}k_{\theta\theta}^{-1} + B_{26}\varepsilon_{\theta\theta}^0 + D_{26}k_{\theta\theta}^1 + E_{26}k_{\theta\theta}^2 + AA_{66}k_{x\theta}^{-1} + B_{66}\gamma_{x\theta}^0 + D_{66}k_{x\theta}^1 + E_{66}k_{x\theta}^2 + F_{66}k_{x\theta}^3$$

$$K_{xx} = D_{11}\varepsilon_{xx}^0 + E_{11}k_{xx}^1 + F_{11}k_{xx}^2 + H_{11}k_{xx}^3 + AAA_{12}k_{\theta\theta}^{-2} + BB_{12}k_{\theta\theta}^{-1} + D_{12}\varepsilon_{\theta\theta}^0 + E_{12}k_{\theta\theta}^1 + F_{12}k_{\theta\theta}^2 + BB_{16}k_{x\theta}^{-1} + D_{16}\gamma_{x\theta}^0 + E_{16}k_{x\theta}^1 + F_{16}k_{x\theta}^2 + H_{16}k_{x\theta}^3$$

$$K_{\theta\theta} = D_{12}\varepsilon_{xx}^0 + E_{12}k_{xx}^1 + F_{12}k_{xx}^2 + H_{12}k_{xx}^3 + AAA_{22}k_{\theta\theta}^{-2} + BB_{22}k_{\theta\theta}^{-1} + D_{22}\varepsilon_{\theta\theta}^0 + E_{22}k_{\theta\theta}^1 + F_{22}k_{\theta\theta}^2 + BB_{26}k_{x\theta}^{-1} + D_{26}\gamma_{x\theta}^0 + E_{26}k_{x\theta}^1 + F_{26}k_{x\theta}^2 + H_{26}k_{x\theta}^3$$

$$K_{\theta z} = BB_{44}k_{\theta z}^{-1} + D_{44}\gamma_{\theta z}^0 + E_{44}k_{\theta z}^1 + F_{44}k_{\theta z}^2 + D_{45}\gamma_{xz}^0 + F_{45}k_{xz}^2$$

$$K_{xz} = BB_{45}k_{\theta z}^{-1} + D_{45}\gamma_{\theta z}^0 + E_{45}k_{\theta z}^1 + F_{45}k_{\theta z}^2 + D_{55}\gamma_{xz}^0 + F_{55}k_{xz}^2$$

$$K_{x\theta} = D_{16}\varepsilon_{xx}^0 + E_{16}k_{xx}^1 + F_{16}k_{xx}^2 + H_{16}k_{xx}^3 + AAA_{26}k_{\theta\theta}^{-2} + BB_{26}k_{\theta\theta}^{-1} + D_{26}\varepsilon_{\theta\theta}^0 + E_{26}k_{\theta\theta}^1 + F_{26}k_{\theta\theta}^2 + BB_{66}k_{x\theta}^{-1} + D_{66}\gamma_{x\theta}^0 + E_{66}k_{x\theta}^1 + F_{66}k_{x\theta}^2 + H_{66}k_{x\theta}^3$$

$$L_{xx} = E_{11}\varepsilon_{xx}^0 + F_{11}k_{xx}^1 + H_{11}k_{xx}^2 + I_{11}k_{xx}^3 + BBB_{12}k_{\theta\theta}^{-2} + DD_{12}k_{\theta\theta}^{-1} + E_{12}\varepsilon_{\theta\theta}^0 + F_{12}k_{\theta\theta}^1 + H_{12}k_{\theta\theta}^2 + DD_{16}k_{x\theta}^{-1} + E_{16}\gamma_{x\theta}^0 + F_{16}k_{x\theta}^1 + H_{16}k_{x\theta}^2 + I_{16}k_{x\theta}^3$$

$$L_{\theta\theta} = E_{12}\varepsilon_{xx}^0 + F_{12}k_{xx}^1 + H_{12}k_{xx}^2 + I_{12}k_{xx}^3 + BBB_{22}k_{\theta\theta}^{-2} + DD_{22}k_{\theta\theta}^{-1} + E_{22}\varepsilon_{\theta\theta}^0 + F_{22}k_{\theta\theta}^1 + H_{22}k_{\theta\theta}^2 + DD_{26}k_{x\theta}^{-1} + E_{26}\gamma_{x\theta}^0 + F_{26}k_{x\theta}^1 + H_{26}k_{x\theta}^2 + I_{26}k_{x\theta}^3$$

$$L_{\theta z} = DD_{44}k_{\theta z}^{-1} + E_{44}\gamma_{\theta z}^0 + F_{44}k_{\theta z}^1 + H_{44}k_{\theta z}^2 + E_{45}\gamma_{xz}^0 + H_{45}k_{xz}^2$$

$$L_{xz} = DD_{45}k_{\theta z}^{-1} + E_{45}\gamma_{\theta z}^0 + F_{45}k_{\theta z}^1 + H_{45}k_{\theta z}^2 + E_{55}\gamma_{xz}^0 + H_{55}k_{xz}^2$$

$$L_{x\theta} = E_{16}\varepsilon_{xx}^0 + F_{16}k_{xx}^1 + H_{16}k_{xx}^2 + I_{16}k_{xx}^3 + BBB_{26}k_{\theta\theta}^{-2} + DD_{26}k_{\theta\theta}^{-1} + E_{26}\varepsilon_{\theta\theta}^0 + F_{26}k_{\theta\theta}^1 + H_{26}k_{\theta\theta}^2 + DD_{66}k_{x\theta}^{-1} + E_{66}\gamma_{x\theta}^0 + F_{66}k_{x\theta}^1 + H_{66}k_{x\theta}^2 + I_{66}k_{x\theta}^3$$

$$Q_{xx} = F_{11}\varepsilon_{xx}^0 + H_{11}k_{xx}^1 + I_{11}k_{xx}^2 + O_{11}k_{xx}^3 + DDD_{12}k_{\theta\theta}^{-2} + EE_{12}k_{\theta\theta}^{-1} + F_{12}\varepsilon_{\theta\theta}^0 + H_{12}k_{\theta\theta}^1 + I_{12}k_{\theta\theta}^2 + EE_{16}k_{x\theta}^{-1} + F_{16}\gamma_{x\theta}^0 + H_{16}k_{x\theta}^1 + I_{16}k_{x\theta}^2 + O_{16}k_{x\theta}^3$$

$$Q_{x\theta} = F_{16}\varepsilon_{xx}^0 + H_{16}k_{xx}^1 + I_{16}k_{xx}^2 + O_{16}k_{xx}^3 + DDD_{26}k_{\theta\theta}^{-2} + EE_{26}k_{\theta\theta}^{-1} + F_{26}\varepsilon_{\theta\theta}^0 + H_{26}k_{\theta\theta}^1 + I_{26}k_{\theta\theta}^2 + EE_{66}k_{x\theta}^{-1} + F_{66}\gamma_{x\theta}^0 + H_{66}k_{x\theta}^1 + I_{66}k_{x\theta}^2 + O_{66}k_{x\theta}^3$$

where

$$\varepsilon_{xx}^0 = \bar{u}_{,x} \qquad k_{xx}^1 = \Psi_{x,x} \qquad k_{xx}^2 = 0$$

$$k_{xx}^3 = -\frac{4}{3h^2}(\bar{w}_{,xx} + \Psi_{x,x}) \qquad k_{\theta\theta}^{-2} = 0$$

$$k_{\theta\theta}^{-1} = \bar{w} + \bar{v}_{,\theta} - R\Psi_{y,\theta} + \frac{R^3(\Delta_2 - \Delta_3)}{\Delta_1}\left(\Psi_{y,\theta} + \frac{\bar{w}_{,\theta\theta} - \bar{v}_{,\theta}}{R}\right)$$

$$\varepsilon^0_{\theta\theta} = \Psi_{y,\theta} - \frac{R^2(\Delta_2 - \Delta_3)}{\Delta_1}\left(\Psi_{y,\theta} + \frac{\bar{w}_{,\theta\theta} - \bar{v}_{,\theta}}{R}\right)$$

$$k^1_{\theta\theta} = \frac{R^2(\Delta_2 - \Delta_3)}{\Delta_1}\left(\Psi_{y,\theta} + \frac{\bar{w}_{,\theta\theta} - \bar{v}_{,\theta}}{R}\right)$$

$$k^2_{\theta\theta} = \frac{\Delta_3}{\Delta_1}\left(\Psi_{y,\theta} + \frac{\bar{w}_{,\theta\theta} - \bar{v}_{,\theta}}{R}\right)$$

$$k^{-1}_{x\theta} = \bar{u}_{,\theta} - R\Psi_{x,\theta} + \frac{4R^3}{3h^2}(\bar{w}_{,x\theta} + \Psi_{z,\theta})$$

$$\gamma^0_{x\theta} = \bar{v}_{,x} + \Psi_{x,\theta} - \frac{4R^2}{3h^2}(\bar{w}_{,x\theta} + \Psi_{x,\theta})$$

$$k^1_{x\theta} = \Psi_{y,x} + \frac{4R}{3h^2}(\bar{w}_{,x\theta} + \Psi_{x,\theta})$$

$$k^2_{x\theta} = -\frac{4}{3h^2}(\bar{w}_{,x\theta} + \Psi_{x,\theta}) + \frac{R\Delta_2}{\Delta_1}\left(\Psi_{y,x} + \frac{\bar{w}_{,x\theta} - \bar{v}_{,x}}{R}\right)$$

$$k^3_{x\theta} = \frac{\Delta_3}{\Delta_1}\left(\Psi_{y,x} + \frac{\bar{w}_{,x\theta} - \bar{v}_{,x}}{R}\right)$$

$$\gamma^0_{xx} = \bar{w}_{,x} + \Psi_x \qquad k^2_{xx} = -\frac{4}{h^2}(\bar{w}_{,x} + \Psi_x)$$

$$k^{-1}_{\theta z} = -\frac{R^3(\Delta_2 - \Delta_3)}{\Delta_1}\left(\Psi_y + \frac{\bar{w}_{,\theta} - \bar{v}}{R}\right) + \bar{w}_{,\theta} - \bar{v} + R\Psi_y$$

$$\gamma^0_{\theta z} = \frac{R^2(\Delta_2 - \Delta_3)}{\Delta_1}\left(\Psi_y + \frac{\bar{w}_{,\theta} - \bar{v}}{R}\right)$$

$$k^1_{\theta z} = \frac{R(\Delta_2 + \Delta_3)}{\Delta_1}\left(\Psi_y + \frac{\bar{w}_{,\theta} - \bar{v}}{R}\right)$$

$$k^2_{\theta z} = \frac{2\Delta_3}{\Delta_1}\left(\Psi_y + \frac{\bar{w}_{,\theta} - \bar{v}}{R}\right)$$

III. Solution Methodology

A. Finite Length Cylinder

In this case, a separated solution is used in the form of a double infinite trigonometric series. For the case of complete fixation at the boundaries, the following form is used:

$$U_i = \sum_{m=0}^{M}\sum_{n=2}^{N}(U_{imn}\sin n\theta + U'_{imn}\cos n\theta) \times \left(\cos\frac{m\pi x}{L} - \cos\frac{(m+2)\pi x}{L}\right) \tag{25}$$

where

$$i = 1, 2, 3, 4, 5$$

$$m = 0, \ldots, M \quad \text{and} \quad n = 2, \ldots, N$$

and

$$U_1 = \bar{u}, \quad U_2 = \bar{v}, \quad U_3 = \bar{w}, \quad U_4 = \Psi_x, \quad U_5 = \Psi_y$$

The terms corresponding to $n = 0$ and 1 are not included because they do not represent shell buckling modes.

The Galerkin procedure is employed in both directions, and the result is $10(M + 1)$ set of homogeneous linear algebraic equations in U_{imn} and U'_{imn} where $m = 0, 1, 2, \ldots, M$, $n = N$ (only one due to the orthogonality of $\sin n\theta$ and $\cos n\theta$), and $i = 1, 2, \ldots, 5$. For the trivial solution to exist, the determinant of the coefficients of the above system must be zero. The lowest eigenvalue represents the critical load. This requires minimization with respect to N.

B. Infinite Length Cylinder

By assuming that the cylindrical shell is very long, the buckling equations, Eqs. (20–24), reduce to a system of five linear ordinary differential equations in the five dependent variables. Because the operators are linear, an equivalent set of three, higher order, linear, ordinary, differential equations in $\bar{u}$, $\bar{v}$, and $\bar{w}$, denoted by $U(y)$, $V(y)$ and $W(y)$ can be obtained.

The following solution satisfies the buckling equations as well as the continuity conditions in the circumferential direction.

$$U = A_n \cos n\theta; \quad V = B_n \cos n\theta; \quad \text{and} \quad W = C_n \sin n\theta \tag{26}$$

Substitution into the derived equivalent set of buckling equations yields three linear, homogeneous, algebraic equations in A_n, B_n, and C_n.

For a nontrivial solution to exist, the determinant of the coefficients must vanish. The lowest eigenvalue represents the critical condition. Minimization with respect to n is required.

In the derivation of the field equations, it is assumed that the pressure load is constant-directional (see Ref. 27).

IV. Applications

Limited numerical results are presented herein for two load cases: uniform axial compression and lateral pressure on a very long cylinder. The material is Boron/Epoxy, and several stacking sequences are used. Numerical results are obtained by employing the CRAY-YMP machine of the Ohio State Supercomputer Center in Columbus, Ohio. The geometric and material properties are

$$R = 19.05 \text{ cm}; \quad R/h = 15, 10; \quad L/R = 1, 5$$

$$E_{11} = 206.844 \times 10^9 \text{ Pa}; \quad \nu_{12} = 0.21$$

$$G_{12} = 4.48162 \times 10^9 \text{ Pa}; \quad E_{22} = 18.6159 \times 10^9 \text{ Pa}$$

$$\nu_{13} = 0.21; \quad G_{13} = 4.48162 \times 10^9 \text{ Pa}$$

$$E_{33} = 18.6159 \times 10^9 \text{ Pa}; \quad \nu_{23} = 0.45$$

$$G_{23} = 2.55107 \times 10^9 \text{ Pa}$$

The stacking sequences are presented in Table 1. The first 14 stackings (1–14) use 0-deg and 90-deg plies (orthotropic shells).

Table 1 Stacking sequences used

Code no.	Stacking sequence	Code no.	Stacking sequence
1	$[0_3]_s$	14	$[90^\circ_2/0_2]_s$
2	$[0^\circ_2/90^\circ]_s$	15	$[45^\circ_2/-45^\circ]_s$
3	$[0^\circ/90^\circ/0^\circ]_s$	16	$[45^\circ/-45^\circ_2]_s$
4	$[0^\circ/90^\circ_2]_s$	17	$[-45^\circ/45^\circ/-45^\circ]_s$
5	$[90^\circ/0^\circ_2]_s$	18	$[-45^\circ_2/45^\circ]_s$
6	$[90^\circ/0^\circ/90^\circ]_s$	19	$[-45^\circ_2/45^\circ_2]_s$
7	$[90^\circ_2/0^\circ]_s$	20	$[45^\circ_2/-45^\circ_2]_s$
8	$[90^\circ_3]_s$	21	$[45^\circ/-45^\circ/45^\circ/-45^\circ]_s$
9	$[0^\circ_2/90^\circ_2]_s$	22	$[45^\circ/-45^\circ_2/45^\circ]_s$
10	$[0^\circ/90^\circ/0^\circ/90^\circ]_s$	23	$[30^\circ_2/-60^\circ]_s$
11	$[0^\circ/90^\circ_2/0^\circ]_s$	24	$[60^\circ_2/-30^\circ]_s$
12	$[90^\circ/0^\circ_2/90^\circ]_s$	25	$[30^\circ_2/-60^\circ_2]$
13	$[90^\circ/0^\circ/90^\circ/0^\circ]_s$	26	$[60^\circ_2/-30^\circ_2]_s$

Table 2 Critical axial compression in (n/m)10^{-6}/wave number (boron/epoxy; R/h = 15)

Stacking code	L/R = 1				L/R = 5			
	CL	FOSD	FOSD W/CF	HOSD	CL	FOSD	FOSD W/CF	HOSD
1	82.66/3	53.76/3	33.62/3	34.50/3	12.52/3	12.71/3	12.56/3	12.45/3
3	66.64/3	35.46/3	29.86/3	32.40/3	15.77/3	14.98/3	14.77/3	14.76/3
6	32.83/3	23.41/3	21.81/4	21.40/4	15.11/3	14.01/3	13.78/3	13.72/3
8	14.71/3	13.82/3	13.34/3	13.22/3	11.38/3	10.67/3	10.54/3	10.54/3
15	28.41/3	18.92/2	17.65/2	17.33/2	14.50/2	12.76/2	12.49/2	12.41/2
16	33.76/2	21.61/2	19.99/2	19.45/2	17.45/2	14.93/2	14.53/2	14.38/2
17	36.56/2	23.42/2	21.63/2	21.23/2	20.66/2	17.06/2	16.48/2	16.12/2
18	28.41/3	18.92/2	17.65/2	17.33/2	14.50/2	12.76/2	12.49/2	12.41/2
23	57.63/2	28.44/2	25.52/2	24.30/2	12.03/3	10.66/3	10.38/3	10.12/3

The next eight (15–22) use ±45-deg plies, while the last four (23–26) use ±30-deg and ±60-deg plies. All laminates are symmetric. In order to achieve different R/h ratios, the radius is kept constant and the thickness is varied. The number of plies shown in Table 1 represents the smallest number of plies employed. As the total thickness increases, the number of plies increases accordingly.

A. Uniform Axial Compression

Results are presented in Table 2 for nine stacking sequences, for one thickness (R/h = 15) and two lengths (L/R = 1 and 5). Critical loads corresponding to classical theory (CL), first order shear deformable theory (FOSD), first order shear deformable theory with a shear correction factor of 5/6 (FOSD-W/CF) and higher order shear deformable theory (HOSD) are presented. Among the important observations, one can list the following:

1) Classical theory yields results that overestimate the buckling strength of the laminated shells, especially for the shorter length.

2) First order shear deformable theory with the correction factor of 5/6 yields extremely accurate predictions, if one assumes that the higher order theory results are indeed accurate.

3) The strongest configuration for L/R = 1 corresponds to $(0^\circ_3)_s$ and the weakest to $(90^\circ_3)_s$.

4) The strongest configuration for L/R = 5 corresponds to $(-45^\circ/45^\circ/-45^\circ)_s$ while the weakest corresponds to $(30^\circ_2/-60^\circ)_s$ with $(90^\circ_3)_s$, a close second.

B. Lateral Pressure ($L/R \to \infty$)

Results for this load case are presented in Table 3 for all 26 stacking sequences and two thicknesses (R/h = 15 and R/h = 10). Critical pressures corresponding to first order shear deformable theory with a shear correction factor of 5/6 are only presented for R/h = 10. Among the most important observations for this load case one can list the following:

1) Classical theory yields critical pressures that are higher than those corresponding to the higher order theory. For R/h = 15, the maximum difference is approximately 10%, whereas for R/h = 10, it becomes 20%.

2) The effect of stacking on the critical pressure is independent of shell theory. The strongest configuration corresponds to $(90^\circ_3)_s$. For this configuration the bending stiffness in the hoop direction is the largest and the results are not surprising. Because of this, stacking sequence 1, $(0^\circ_3)_s$, is the weakest configuration.

3) First-order shear deformable theory yields critical pressures that are very close to those corresponding to the higher order theory. The difference is smaller than 1% for most stacking sequences.

Calculations show that the buckling pressures are very high. A question then arises, as to whether collapse failure is caused by strength or by buckling. In checking the thicker geometry (R/h = 10), it was established that failure will occur by buckling before first ply failure by strength is recorded. The strength values are taken from Table 5.1 of Ref. 28.

Table 3 Critical pressure in Pa × 10^{-3} (boron/epoxy)

Stacking code	R/h = 15			R/h = 10			
	CL	FOSD	HOSD	CL	FOSD	FOSD W/CF	HOSD
1	1,847	1,771	1,765	6,274	5,791	5,791	5,791
2	2,537	2,433	2,426	8,618	7,997	7,929	7,929
3	6,687	6,274	6,191	22,614	20,063	19,788	19,512
4	7,377	6,943	6,881	24,959	22,270	21,994	21,925
5	14,982	13,527	13,286	50,607	41,506	40,403	39,920
6	15,671	14,286	14,196	52,952	44,264	43,230	43,644
7	19,822	17,788	17,354	66,948	54,193	52,607	51,573
8	20,512	18,567	18,333	69,223	57,088	55,572	55,641
9	4,185	3,978	3,957	14,134	12,962	12,824	12,824
10	7,680	7,198	7,136	25,924	23,028	22,683	22,614
11	9,432	8,770	8,639	31,853	27,786	27,300	26,889
12	12,934	11,852	11,810	43,644	36,956	36,128	36,680
13	14,679	13,362	13,203	49,573	41,299	40,334	40,334
14	18,181	16,313	15,899	61,363	49,642	48,194	47,160
15	6,226	5,805	5,743	21,029	18,478	18,202	17,995
16	6,226	5,839	5,791	21,029	18,684	18,478	18,340
17	6,226	5,853	5,826	21,029	18,753	18,547	18,615
18	6,226	5,805	5,743	21,029	18,478	18,202	17,995
19	6,226	5,819	5,750	21,029	18,547	18,271	18,064
20	6,226	5,819	5,750	21,029	18,547	18,271	18,064
21	6,226	5,853	5,819	21,029	18,753	18,547	18,547
22	6,226	5,860	5,833	21,029	18,822	18,615	18,615
23	3,144	2,999	2,985	10,617	9,790	9,721	9,652
24	11,790	10,776	10,576	39,782	33,508	32,750	32,198
25	3,964	3,771	3,750	13,444	12,272	12,203	12,134
26	10,960	10,025	9,838	37,025	31,164	30,475	29,992

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